The Method of Interlacing Families

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Outline

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1. Introduction to interlacing families
2. Application to construct infinite families of bipartite Ramanujan graphs

If time permits:

Let A be a bounded operator. Then there exists a signing of A s.t

signing

$$\underbrace{\|A \circ S\|_2}_{\text{entry-wise product}} < 2\|A\|_{\ell_\infty}$$

$$\begin{pmatrix} A & 0 \\ 0 & A^T \end{pmatrix}$$

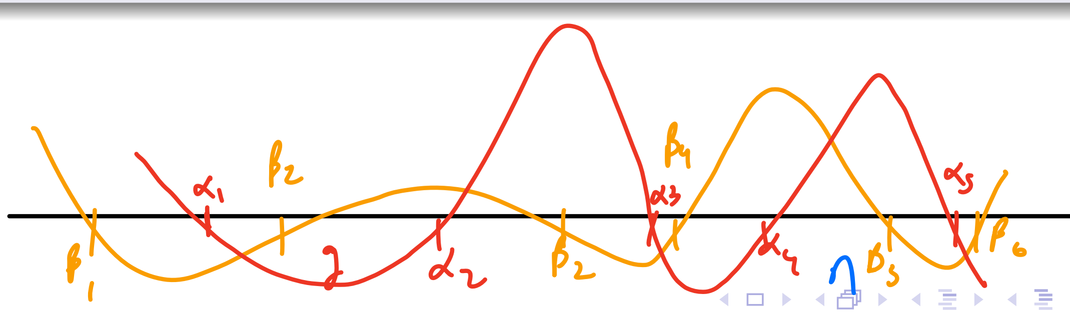
Polynomials with a Common Interlacing

Definition

A polynomials $g(x) = \prod_{i=1}^{n-1}(x - \alpha_i)$ *interlaces* another polynomial $f(x) = \prod_{i=1}^{n}(x - \beta_i)$ if

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} \leq \beta_n$$

Polynomials $f_1, \ldots, f_k$ have a *common interlacing* if there exists a polynomial g that interlaces every f_i



Roots of Polynomials with a Common Interlacing

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Lemma

Let $f_1, \ldots, f_k$ be real-rooted polynomials of the same degree and have positive leading coefficients. We define

"average"

$$f_\emptyset = \sum_{i=1}^{k} f_i \tag{1}$$

If $f_1, \ldots, f_k$ have a common interlacing, then there exists i such that $\lambda_{\max}(f_i) \leq \lambda_{\max}(f_\emptyset)$.

Lemma holds for the k^{th} largest roots
The takeaway here is that when the polynomials have a common interlacing, "averaging" component wise behaves well with respect to the roots. This is not true in general, very easy counterxamples!

Roots of Polynomials with a Common Interlacing - Proof sketch

largest root of g

Proof.

- f_i is positive for large enough x, f_i has *exactly one root* $\geq \alpha_{n-1}$
  - f_i non-positive at α_{n-1}

$\square$

Roots of Polynomials with a Common Interlacing - Proof sketch

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  - f_i non-positive at α_{n-1}
- $f_\emptyset$ non-positive at α_{n-1} and then eventually positive
  - $f_\emptyset$ has one (and hence largest) root $\geq \alpha_{n-1}$ say β_n

□

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  - $f_\emptyset$ has one (and hence largest) root $\geq \alpha_{n-1}$ say β_n
- $0 = f_\emptyset(\beta_n) = \sum_{i=1}^{k} f_i(\beta_n) \implies \exists i : f_i(\beta_n) \geq 0$
- Largest root of f_i is between α_{n-1} and β_n

$\square$

largest root of $f_\emptyset$

Interlacing Families

Definition (Interlacing Family)

Let $S_1, \ldots, S_m$ be finite sets. Suppose that for every assignment $s_1, \ldots, s_m \in S_1 \times \cdots \times S_m$, $f_{s_1}, \ldots, \cancel{f_{s_m}}$ are real-rooted polynomials, degree n polynomials with positive leading coefficients.

Now for every partial assignment $s_1, \ldots, s_k \in S_1 \times \cdots \times S_k$ with $k < m$ define

$$f_{s_1, s_2, \ldots, s_k} = \sum_{s_{k+1} \in S_{k+1}, \ldots, s_m \in S_m} f_{s_1, \ldots, s_k, s_k+1, \ldots, s_m}$$

conditional expection

And finally

$$f_\emptyset = \sum_{s_1 \in S_1, \ldots, s_m \in S_m} f_{s_1, \ldots, s_m}$$

$\{f_{s_1, \ldots, s_m}\}$ form an interlacing family if for all k and every partial assignment: $\{f_{s_1, \ldots, s_k, t}\}_{t \in S_{k+1}}$ have a common interlacing

$S_i = \{\pm 1\} \quad , i \in \{1, 2\}$

λ_{max}

$f_\phi = \underline{f_{1,k}} + \underline{f_{-1,k}}$

$\leq \lambda_{max}$

$f_{1,k} = \overbrace{f_{1,1} + f_{1,-1}}$

s_1

common interlacing

$f_{-1,k} = f_{-1,1} + f_{-1,-1}$

$\leq \lambda_{max}$

$f_{1,1}$

$f_{1,-1}$

$f_{-1,1}$

$f_{-1,-1}$

Roots of Interlacing Families

Theorem

Let $S_1, \ldots, S_m$ be finite sets and $\{f_{s_1,\ldots,s_m}\}$ be an interlacing family. Then there exists some assignment $s_1, \ldots, s_m \in S_1 \times \cdots \times S_m$ such that $\lambda_{\max}(f_{s_1,\ldots,s_m}) \leq \lambda_{\max}(f_{\emptyset})$

Real-rootedness criterion

Checking if $f_1, \ldots, f_k$ have a common interlacing is difficult in a lot of cases

Lemma

Let $f_1, \ldots, f_k$ be polynomials of the same degree with positive leading coefficients. Then $f_1, \ldots, f_k$ have a common interlacing if and only if all convex combinations of $f_1, \ldots, f_k$ are real rooted

$$\sum_{i=1}^{k} \lambda_i f_i \ \textit{real-rooted} \quad \forall \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1$$

Convex combinations real rooted $\Rightarrow$

Structure of a Proof Using Interlacing Families

Assuming we have a family $\{f_{s_1,\ldots,s_m}\}$ of real-rooted, degree n polynomials with positive leading coefficients:

1. Prove that the family is indeed an interlacing family. This often amounts to the problem of proving real-rootedness of convex combinations

Structure of a Proof Using Interlacing Families

Assuming we have a family $\{f_{s_1,\ldots,s_m}\}$ of real-rooted, degree n polynomials with positive leading coefficients:

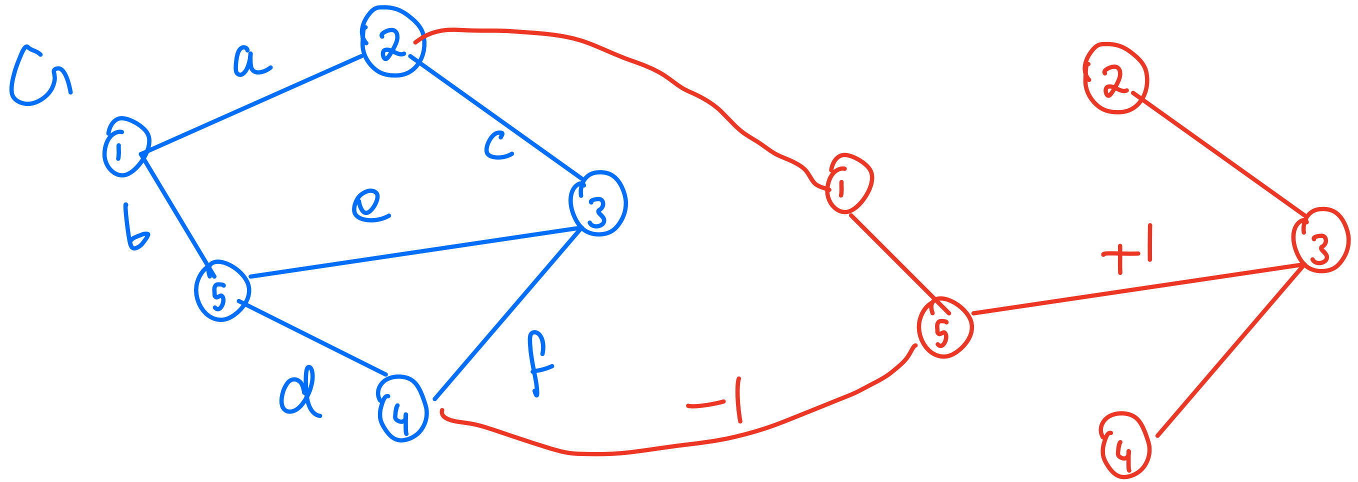
1. Prove that the family is indeed an interlacing family. This often amounts to the problem of proving real-rootedness of convex combinations
2. Bound the "average" polynomial $f_\emptyset$

Interlacing Families 1: Bipartite Ramanujan Graphs

$-d$ $\qquad\qquad\qquad$ d

Goal: Construct infinite families of bipartite d-regular graphs with *non-trivial eigenvalues* bounded by $2\sqrt{d-1}$.
Idea: Start with a d-regular bipartite Ramanujan graph G with n vertices and m edges and construct a 2-lift of G whose eigenvalues remain bounded by $2\sqrt{d-1}$

2-Lifts and Signings



We make a copy of the graph. Then for every edge in the copy, we have 2 choices either leave the edge connecting the copied vertices or cross them with vertices in the original graph.
Sign each edge ± 1 to denote whether we crossed or not.
Can we find a signing s.t 2-Lifted graph has $\lambda \leq 2\sqrt{d-1}$

A_s is obtained simply obtained by assigning ± 1 appropriately to each edge entry of A.

Theorem (Eigenvalues of 2-Lifts)

Let A be the adjacency matrix of a graph G and A_s be the sign matrix of some 2-lift of G, then the eigenvalues of the 2-lifted graph is the union of the eigenvalues of the original graph and the eigenvalues of A_s

Can we always find a 2-lift such that the eigenvalues of A_s are bounded by $2\sqrt{d-1}$?

Applying the method of interlacing families:

$$f_s = \det(xI - A_s) \tag{2}$$

We have a real-rooted, degree n polynomial with positive leading coeffiecient for each signing of G

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A_S = \begin{pmatrix} 0 & \pm 1 & 0 & 0 \\ \pm 1 & 0 & \pm 1 & 0 \\ 0 & \pm 1 & 0 & \pm 1 \\ 0 & 0 & \pm 1 & 0 \end{pmatrix}$$

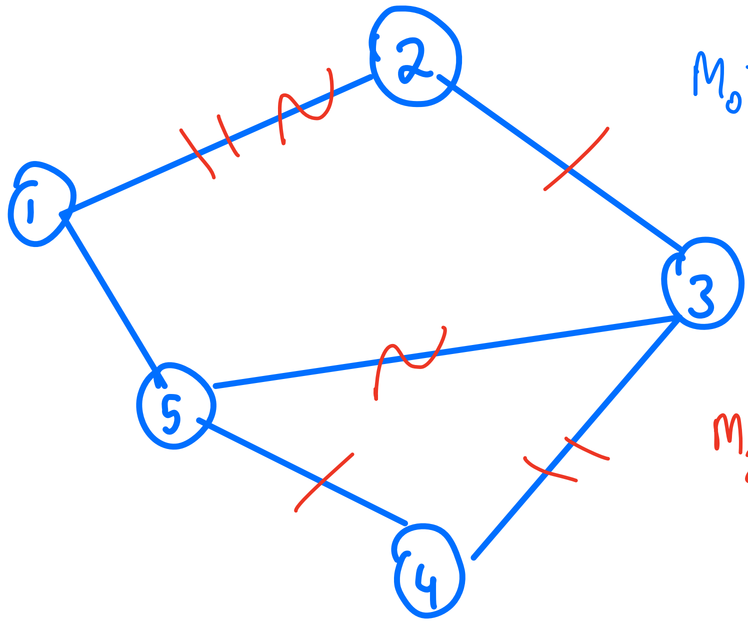
The Matching Polynomial

Definition

Let G be a graph with n vertices then $\mu_G = \sum_{i>0} x^{n-2i}(-1)^i m_i$ is its matching polynomial.
Where m_i is the number of matchings of G with i edges. A matching of a graph is any subset of edges, such that no vertex is touched more than once.

Theorem

Let G be a graph, then $\mu_G(x)$ is real-rooted.
Let G have maximum degree d, then all roots of μ_G have absolute value bounded by $2\sqrt{d-1}$



$m_0 = 1$ (Trivial matching with no edges)

$m_1 = 6$ (Just take any of the 6 edges)

$m_2 = 3$

$$\Rightarrow \mu_G(x) = x^4 - 6x^2 + 3$$

Bounding the Average with μ_G

Roots of μ_G bounded by $2\sqrt{d-1}$ $\Rightarrow$ If we show f_s is an interlacing family we are done!

Theorem

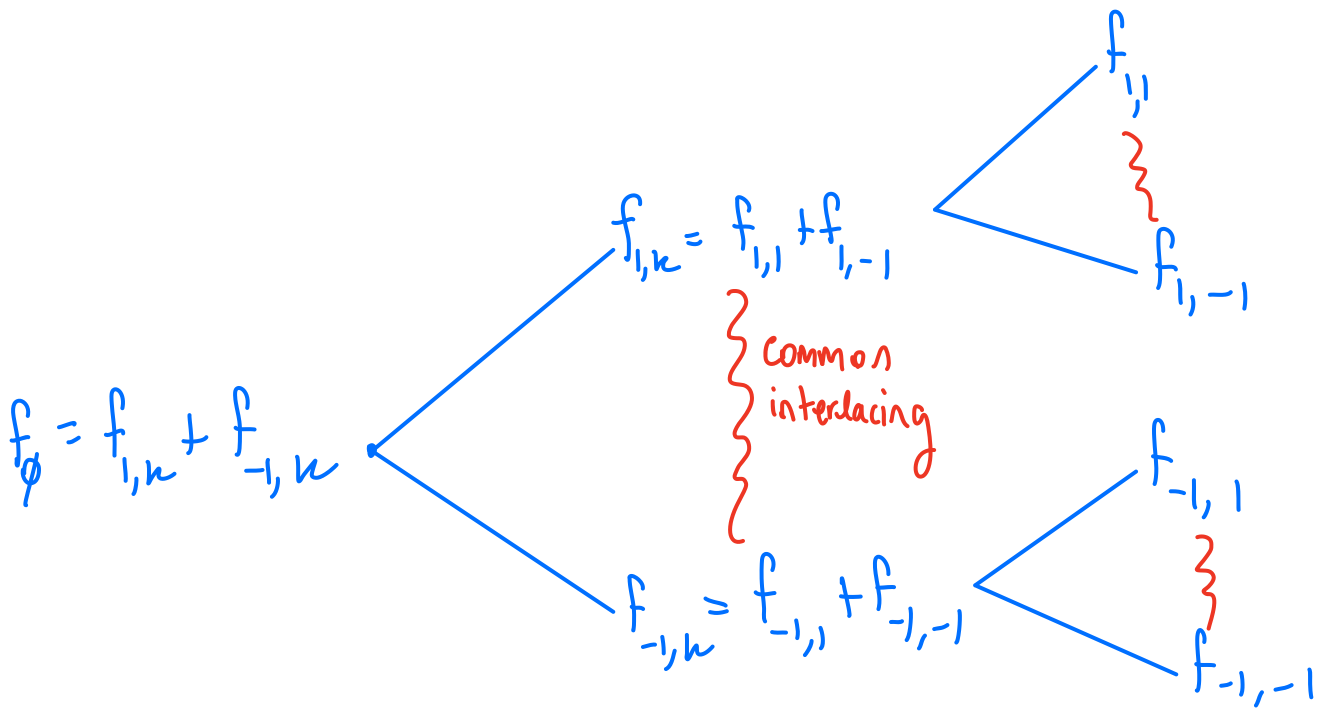
Let G be a graph with n vertices and m edges and f_s be defined as above, then

$$f_\emptyset = \mathbb{E}_{s\in\{\pm 1\}^m}[f_s(x)] = \mu_G(x)$$

$$= \mathbb{E}_s[\det(xI - A_s)]$$

$$= \mathbb{E}_s[\text{sum over permutations}]$$

$$= \text{sum } \mathbb{E}_s(\text{permutations})$$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Definition (Real Stability)

A multivariate polynomial $f \in \mathbb{R}[z_1, \dots, z_n]$ is real stable if $f \equiv 0$ or if

$$\Im(z_i) > 0 \forall i \implies f(z_1, \dots, z_n) \neq 0$$

Closure under:

1. Scaling $f \mapsto cf(a_1 z_1, \dots, a_n z_n), c \in \mathbb{C}, a \in \mathbb{R}^n$
2. Specialization: $f \mapsto f(a, z_2, \dots, z_n), a$ with $\Im(a) \geq 0$
3. Differentiation $f \mapsto \partial_1 f$

We need to prove that nodes with a common parent have a common interlacing

$\Longleftrightarrow$

Prove that all convex combinations of these nodes are real-rooted

Show that they are images of real-stable polynomials under operations that preserve real stability

Show $\lambda f_{s_1,\dots,s_k,+1}(x) + (1-\lambda) f_{s_1,\dots,s_k,-1}(x)$ is real-rooted $\forall \lambda \in [0,1]$ and all partial assignments $s_1,\dots,s_k$

$s_1,\dots,s_k$ are fixed ; s_{k+1} is $+1$ with prob λ &
-1 with prob $(1-\lambda)$
; $s_{k+2},\dots s_m$ are $\underbrace{\text{uniformly } \pm 1}_{\text{prob } 1/2 \text{ each}}$

Essentially proving that all of these conditional expectations are real rooted.

So if we prove $\mathbb{E}_{s\in\{\pm1\}^m}[f_s(x)]$ is real rooted for any independent distribution of signings we are done since the special conditional expectations are of this form

Note that we already know this for $f_\phi = \mathbb{E}_s[f_s]$ where all $s_1,\dots,s_m$ were picked uniformly $\pm 1/2$, so we are "generalizing" this for any distribution.

Idea: Show f_s are the images of real-stable polynomials under operations that preserve real-stability

MSS showed that $\mathbb{E}[\underset{\text{char poly.}}{\chi}(\sum_i a_i a_i^*)]$ is real rooted for $a_1, \dots, a_n$ independently chosen random vectors

$\mathbb{E}[\chi(\sum a_i a_i^*)]$ is the mixed characteristic polynomial of $a_1, \dots, a_n$.

How to bring $\sum_{s \in \{\pm 1\}^m} f_s(x) = \sum_{s \in \{\pm 1\}^m} \chi(A_s)$ to the form of a mixed characteristic polynomial?

Need to express A_s as the sum of random rank 1 matrices

For each edge in our graph we add

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{sign } +1} \quad \text{or} \quad \underbrace{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}_{\text{sign } -1} \quad \text{to } A_s$$

But these are rank 2! To fix this we add 1s along the diagonal, so we instead add:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \end{pmatrix}$$

So we have $A_s + D = \sum a_i a_i^T$ where $a_i = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $a_i = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ independently.